Interaction of a Gravitational Wave and a Superconductor

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Abstract

I have reviewed some of the work done describing the interaction of a superconductor and a gravitational wave. There have been proposals[1, 2, 3] that they could be used to build better gravitational wave detectors. To begin, I have summarized a paper by B. DeWitt on gravitational drag effect on superconductors[4]. Then I have reviewed the work of H. Peng et. al. on the electrodynamics of moving superconductors[2]. I have then briefly described the paper by R. Chiao on using superconductors as GW transducers[3]. Finally, I have summarized the results of A. Licht[5] on the same topic, which concludes that the predicted effects are extremely small and no different from those of normal conductors.

Introduction

This paper is primarily motivated by the recent proposals that Superconductors could be used as detectors of Gravitational Waves.

Gravitational waves (GW) are a prediction of the General theory of Relativity. They appear as a solution to the linearized Einstein equations in free space[6]. They can be considered as ripples in space-time (metric) which propagate away at the speed of light from a massive source (which has a changing quadrupole or higher mass moment).

The subject of gravitational wave detection became an experimental science with the pioneering work of Weber in 1960's when he built the first gravitational wave antenna[7]. They were, however, several orders of magnitude less sensitive than the expected strength of gravitational wave signals. The gravitational wave detectors (GWD) of today can be classified into two kinds

1. Resonant Mass Antenna.

2. Laser Interferometer Antenna.

The primary effect of a passing transverse gravitational wave can be thought of as a force which tends to stretch an object in one direction while simultaneously squeezing in the other[8]. A passing gravitational wave of suitable frequency should excite the fundamental longitudinal mode of a metallic bar when it is suspended and isolated from vibration noise. This excitation can then be detected by using suitable transducers. This is the principle of all Resonant Mass Antennae.

The second kind of GWD use a Michelson type interferometer to measure the length changes between two suspended mirrors, when a GW passes through them.

What makes these waves so very interesting is the fact that they couple very weakly to all matter and can thus carry information from very large distances in time and space. Unfortunately, it also makes them very hard to detect. As a rough estimate, the gravitational wave produced from the collapse of 10 solar mass Black hole binaries at a distance of 20 mega parsecs[9], will cause a length change of 10^{-21} m in a 1 meter long bar of Aluminum!!

There have been suggestions for a long time now that superconductors could be used to directly detect GWs[1, 2, 3]. The hope is that a passing gravitational wave would affect the superelectrons in a superconductor differently and create measurable Electric and Magnetic fields.

Gravitational drag effect on superconductors

The effect of a weak gravitational field on a superconductor was first analyzed by B. DeWitt[4] 1966. He considered the effect of a Lense-Thirring field or the frame dragging effect of a rotating mass on a superconductor. Starting with the lagrangian for a single electron in curved space time,

$$L = -m(-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu})^{\frac{1}{2}} + eA_{\mu}\dot{x}^{\mu} \tag{1}$$

(where m is the mass of the electron $g_{\mu\nu}$ is the space-time metric, e is the electon charge and A_{μ} is the magnetic vector potential)

the Hamiltonian was found to be,

$$H = (g^{ij}g_{0i}g_{0j} - g_{00})^{\frac{1}{2}}[m^2 + g^{kl}(p_k - eA_k)(p_l - eA_l)]^{\frac{1}{2}} - g^{ij}(p_i - eA_i)g_{0j} - eA_0 \tag{2}$$

For the limit of weak fields ($h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}, \eta^{\mu\nu} = diag(-1,1,1,1)$, $|h_{\mu\nu}| << 1$) and small velocities ($v << c$), this reduces to

$$H = -\frac{1}{2m}(\vec{p} - \vec{D})^2 + V \tag{3}$$

where

$V = -eA_0 - \frac{1}{2}mh_{00}$, is the new scalar potential and, $\vec{D} = e\vec{A} + m\vec{h}_0$ acts as the modified vector

potential with $\vec{h}_0 = (h_{01}, h_{02}, h_{03})$ \hfill (4)

Next, he generalized the above Hamiltonian to the ensemble of electrons in a superconductor by modifying the momentum term as above. According to the author, the BCS theory can then be applied to this Hamiltonian resulting in a modified Meissner effect which implies the vanishing of the vector

$$\vec{G} = \vec{\nabla} \times \vec{D} = e\vec{B} + m\vec{\nabla} \times \vec{h}_0 \tag{5}$$

and the vector

$$\vec{F} = -\vec{\nabla}V = e\vec{E} + \frac{1}{2}m\vec{\nabla}h_{00} \tag{6}$$

To give an example of the effect, he then considers a simple gedanken experiment where a circular superconducting ring surrounds a concentric, axially symmetric, quasi-rigid mass at rest.

If the magnetic field was originally zero, then so is $\vec{G}$. If the mass is then spun up to a constant angular velocity, it produces a Lense-Thirring field (or a $\vec{\nabla} \times \vec{h}_0$). As the flux of $\vec{G}$ through the superconducting ring is conserved, this creates a magnetic field to oppose the Lense-Thirring field. The author then estimates the magnitude of the current which produces this magnetic field to be (in SI units) [10]

$$I \approx n \frac{8\pi\varepsilon_0 GmMV}{ed} \tag{7}$$

where G is the Gravitational constant, m and e are the mass and charge of the electron, M is the mass of the rotating body, V is the rim velocity, d is the diameter and n is the number of turns of the superconducting coil. If we use M = 10 kg, $\frac{V}{d} = 2\pi \times 100 \frac{rad}{s}$, and n = 1000, we get

$$I \approx 5 \times 10^{-25} Amps$$

which is 10 orders of magnitude below what can be detected by a SQUID[10].

Electrodynamics of a moving superconductor

In 1991, H. Peng and D. G. Torr tried to generalize the Ginzberg Landau (GL) equation to moving superconductors and derived the internal Electric and Magnetic fields created when superconductor is acted upon by external forces. The analysis however, is very simplistic and makes some assumptions which may not be true.

The authors started by using the expression for the supercurrent density in GL theory

$$\vec{j} = \frac{e_e \hbar}{2im_e}(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) - \frac{e_e^2}{m_e}\psi^* \psi \vec{A} \tag{8}$$

where ψ is the GL order parameter and **A** is the magnetic vector potential. The subscript e refers to the superelectron. They assumed that the forces acting on the superconductor are sufficiently

small so as at keep the magnitude of the order parameter constant. Then the current density expression simplifies to the London equation

$$\vec{j} = -\frac{n e_e \hbar}{m_e}\vec{\nabla}\theta - \frac{2 n e_e^2}{m_e}\vec{A} \qquad (9)$$

This is equivalent to

$$\frac{\partial \vec{v}_e}{\partial t} = \frac{e}{m_e}\vec{E}$$

and

$$\vec{\nabla}\times\vec{v}_e = -\frac{e}{m_e}\vec{B}$$

These equations can be combined into a single covariant form,

$$\frac{\partial p_\mu}{\partial x^\nu} - \frac{\partial p_\nu}{\partial x^\mu} = 0 \qquad \text{where } \mu, \nu \text{ run from 0 to 3} \qquad (10)$$

and $p_\mu = m_e u_\mu + e A_\mu$

Here, u_μ is the superelectron four-velocity.

In the presence of an external nonelectromagnetic force $\mathbf{f}_e$ acting on the superconductor, they generalize the momentum to

$$p_\mu = m u_\mu + e A_\mu - \int K_\mu d\tau \qquad (11)$$

where $d\tau = dt\sqrt{(1-\beta_e^{\,2})}$ and $\vec{f}_e = \vec{K}\sqrt{(1-\beta_e^{\,2})}$

It is not clear whether this can be applied to the case of force due to a gravitational wave.

The net four-current density is

$$J_\mu = J_{e\mu} + J_{i\mu} = 2 n e (u_\mu - U_\mu) \qquad (12)$$

where $J_{e\mu}$ and u_μ are the superelectron current density and four-velocity whereas $J_{i\mu}$ and U_μ are

those of the ions.

According to the authors, Equations (11) and (12) together with Maxwell's equations 'form a complete set of covariant equations which govern the electrodynamics of an arbitrarily moving superconductor and a superconductor under the influence of external force'. It is perhaps presumed that the complete solutions will be found on using the boundary conditions which are determined by the geometry of the superconductor.

In the limit of low velocities, these equations reduce to the following three

$$\vec{\nabla}^2 \vec{J} = \frac{1}{\lambda^2}\left[\vec{J} + \frac{m_e}{\mu_0 e}\left[\vec{\nabla}\times\vec{\nabla}\times v_i + \frac{\partial^2 v_i}{\partial t^2} \right] - \frac{1}{\mu_0 e}\left[\int \vec{\nabla}\times\vec{\nabla}\times f_e dt + \frac{\partial f_e}{dt} \right] \right] \tag{13}$$

$$\vec{\nabla}^2 \vec{E} = \frac{1}{\lambda^2}\left[\vec{E} - \frac{m_e}{e}\frac{\partial v_i}{\partial t} + \frac{1}{e}f_e \right] \tag{14}$$

$$\vec{\nabla}^2 \vec{B} = \frac{1}{\lambda^2}\left[\vec{B} + \frac{m_e}{e}\vec{\nabla}\times v_i - \frac{1}{e}\int \vec{\nabla}\times f_e dt \right] \tag{15}$$

where

$$\lambda = \left[\frac{m_e}{2\mu_0 ne^2} \right]^{\frac{1}{2}}$$, v_i is the velocity of the ions and **B** is the net magnetic field.

The solutions to the above three equations are of the form

$$\vec{J} = \vec{J}_{screen} + \vec{J}_{interior}$$

$$\vec{B} = \vec{B}_{screen} + \vec{B}_{interior}$$

$$\vec{E} = \vec{E}_{screen} + \vec{E}_{interior}$$

Only the interior fields and current densities are studied and the authors restrict themselves to cases with $\vec{\nabla}^2 \vec{J}_{int} = \vec{\nabla}^2 \vec{E}_{int} = \vec{\nabla}^2 \vec{B}_{int} = \vec{\nabla}\times \vec{f}_e = 0$

Then the solutions for the interior are

$$\vec{J}_{int} = \frac{1}{\mu_0 e}\frac{\partial \vec{f}_e}{\partial t} - \frac{m_e}{\mu_0 e}\frac{\partial^2 \vec{v}_i}{\partial t^2} - \frac{m_e}{\mu_0 e}\vec{\nabla}\times\vec{\nabla}\times v_i \tag{16}$$

$$\vec{E}_{int} = \frac{m_e}{e}\frac{\partial \vec{v}_i}{\partial t} - \frac{\vec{f}_e}{e} \tag{17}$$

$$\vec{B}_{int} = -\frac{m_e}{e}\vec{\nabla}\times\vec{v}_i \tag{18}$$

Some simple cases are then considered. I will list two of them

a) Uniformly accelerated superconductor

$$\frac{\partial \vec{v}_i}{\partial t} = const = a$$

Then the solutions are

$$\vec{J}_{int} = 0 \tag{19a}$$

$$\vec{B}_{int} = 0 \tag{19b}$$

$$\vec{E}_{int} = -\frac{m_e}{|e|}a \tag{19c}$$

For a 2m bar, with a = 10 m/s^2, the voltage difference at the ends of the bar is about 10^{-10} volts, which is measurable.

b) A superconductor in the presence of a GW

For this case,

$$\vec{f}_e = m_e\vec{a}_{GW}, \quad \frac{\partial \vec{f}_e}{\partial t}\neq 0, \quad \frac{\partial \vec{v}_i}{\partial t}\neq 0, \quad \frac{\partial^2 \vec{v}_i}{\partial t^2}\neq 0 \text{ and } \vec{\nabla}\times\vec{v}_i = 0$$

The equation of motion for the ions then becomes,

$$\frac{d^2\vec{x}_i}{dt^2} + \frac{1}{\tau_0}\frac{d\vec{x}_i}{dt} - \omega_0^2\vec{x}_i = \vec{a}_{GW} - \frac{e\vec{E}}{m_i} \tag{20}$$

where τ_0 is the damping time and ω_0 is the resonant frequency of the bar whereas ω is the
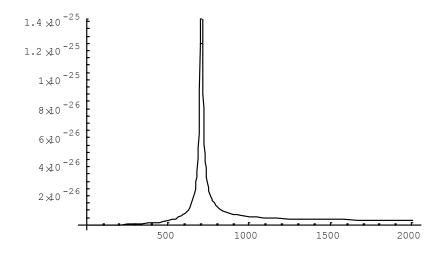
frequency of the wave.

Solving for the Electric field, we get

$$\vec{E}_{int} = \frac{m_e}{e} \frac{\omega_0^2 - i\dfrac{\omega}{\tau}}{2\omega^2 - \omega_0^2 + i\dfrac{\omega}{\tau}} \vec{a}_{GW} \tag{21}$$

Based on this result, the author proposes a new gravitational wave antenna. However, there are no estimates of the magnitude of this effect.

Assuming bar of length $l = 1$ m,

$\omega_0 = 1000 Hz$, $\tau = 1000s$ and $|\vec{a}_{GW}| \approx \omega^2 hl = \omega^2 10^{-21}$

a graph of Electrics field (V/m) is plotted as a function of the frequency (Hz).

At $\omega = \dfrac{1000}{\sqrt{2}} Hz$ we get $\vec{E}_{int} \approx 10^{-21} V/m$. While this is quite small, it is much larger than expected and is several orders of magnitude larger than the value estimated by Licht[5]. It is likely that the way the force due to the gravitational wave is treated is incorrect and a fully general relativistic treatment is necessary.

While there is certainly a force on the superelectrons due to the GW, there is no motivation here to suspect any enhancement or amplification effect. In fact, as Licht[5] later proves, this is one of

the poorer ways of measuring GW as the efficiency of conversion from gravitational to EM energy is extremely small.

Superconductors as GW transducers

In 2002 Raymond Chiao[3] proposed the use of superconductors as Gravitational wave Transducers and conducted a simple Hertz like experiment to test his hypotheses. He gave a number of arguments why a superconductor should be a much better transducer as compared to a mechanical system like a bar.

One of his main arguments is that the quantum systems that exhibit long range 'quantum rigidity' like a superconductor, should couple much more efficiently to a gravitational wave. Due to the classical coupling via acoustic waves, classical matter used as an antenna for gravitational waves is restricted to roughly the size λ_s (wavelength of sound). In the quantum coupling, this restriction may not apply. Since the macroscopic quantum coherences in quantum fluids can in principle be of the same length scale as the gravitational wavelength λ, it is suggested that one could in principle, because the length restrictions do not apply in quantum fluids, replace the speed of sound v_s in the material by the speed of light c.

Since the radiation emission efficiency is proportional to $(v_s)^4$ for a cylindrical bar, Chiao's argument suggests an increased radiation emission efficiency of $(c/v_s)^4 \sim 10^{20}$. It should be noted here that, in contrast to the classical considerations of gravitational-wave detection, where (astrophysical) frequencies in the range of Hz to kHz are considered, the frequencies, corresponding to the above arguments of wavelengths on the same length scale as the detector,

would be of the order of 10 MHz and higher. No detectable astrophysical sources are available in this frequency range.

Another argument is that while in classical materials, the gravitational 'resistance' is generally much higher than Z_g gravitational impedance of free space, leading to virtual transparency with respect to the gravitational wave, quantum materials such as superconductors could have a 'resistance' comparable to that of Z_g which could lead to maximal absorption of the gravitational wave. Both these arguments have been questioned for using classical results and applying them directly to quantum systems[10].

He then argues that the gravitomagnetic field created by a passing gravity wave is expelled from the interior of a superconductor and goes on to try to estimate the coupling energy of the interaction. However, while he does propose a way of doing this, he does not carry out the calculation to the end. He then focuses on explaining the use of extreme Type 2 Superconductors for natural impedance matching. Towards the end of the paper, he explains a Hertz like experiment he carried out to test his theory.

Two high temperature superconductor samples were placed inside Faraday cages. A 12 GHz microwave source inside one provided a source of EM waves incident on sample A. A microwave detector inside the other cage was set up to detect any EM waves emitted from sample B. The cages prevented direct EM coupling. The hope was that, if the superconductors acted as transducers between EM and gravitational radiation of sufficient efficiency, an EM to GW conversion would occur in sample A. Since GWs couple very weakly to ordinary matter, any GW produced would propagate to sample B where the reverse GW to EM process would occur. Unsurprisingly, no signal was detected.

The main reason cited for the negative result is that the superconductor samples may be attenuating the microwaves before they can penetrate to a depth at which they are maximally impedance-matched. This attenuation may occur via absorption by non-superconducting electrons in the sample. This observed process may be particular to the choice of high temperature superconductor. The authors note that an improved experiment using a low temperature superconductor may give different results.

Gravitational waves on conductors

In a paper in March 2004, A. Licht[5] carried out a thorough analysis of the effect of gravitational waves on conductors including superconductors. I will try to summarize some of his methods and his results.

He started by constructing the Fermi normal coordinates for a superconductor and deriving the expressions for the metric perturbations in the presence of a gravitational wave.). The author did not take into account the spin of the electron.

Then, using the Lagrangian for a charged particle in curved space (expression (1)), and linearizing for weak fields and small velocities, the Hamiltonian (3) is derived. The field equations inside a conductor are of the form,

$$\nabla^2 \phi - \kappa^2 \phi = f(\vec{x}) \qquad (22)$$

where κ is the inverse of the skin depth, for normal conductors, or the London penetration depth λ for superconductors, φ is the scalar potential and $f(\vec{x})$ is the source proportional to the gravitational field.

Assuming that the wavelength of the GW is much larger than the dimensions of the

superconductor which is much larger than the penetration depth, the general solution reduces to,

$$\phi(\vec{x}) = -\frac{f(\vec{x})}{\kappa^2} + \phi_H e^{\kappa\zeta} + O\left(\frac{\lambda}{\lambda_W}\right) \tag{23}$$

where ζ denotes the distance from the conductor surface and ϕ_H is the homogenous solution and

is independent of ζ and λ_W is the wavelength of the GW.

The amplitude for the homogenous solution is determined by fitting the interior solution to an

outgoing wave at the boundary. The Electric and magnetic field inside a normal conductor are

then determined by using,

$$\vec{J}_n = \sigma_n\left[\vec{E} - i\omega\frac{mc}{e}\vec{h} - \frac{mc^2}{2e}\vec{\nabla}h_{00}\right] \tag{24}$$

and Maxwell's equations, giving,

$$\vec{E} = i\omega\frac{mc}{e}\vec{h} + \frac{mc^2}{2e}\vec{\nabla}h_{00} + \vec{E}_H e^{\kappa\zeta} \tag{25}$$

where the subscript H refers to the solution to the Homogenous equation. For a superconductor,

the author uses a modified form of the GL Hamiltonian

$$H_S = \frac{1}{2m^*}\left|(-i\hbar\vec{\nabla} - e^*\vec{A} - m^*c\vec{h})\psi\right|^2 + \left(e^*\Phi - \frac{m^*c^2}{2}h_{00}\right)|\psi|^2 - \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \tag{26}$$

and the boundary condition that the superelectron current is parallel to the surface. He then

derives the expression for the order parameter by linearising the GL equation by assuming small

changes in the order parameter.

Finally, the expression for the B and E field is derived using Maxwell's equations,

$$\vec{B} = -\frac{m^*c}{e^*}\vec{\nabla}\times\vec{h} + \vec{B}_H e^{\kappa\zeta} \tag{30}$$

$$\vec{E} = i\frac{\omega}{e^*}(\hbar\vec{\nabla} - m^*c\vec{h}) - i\omega\lambda_L^2\vec{\nabla}\times\left(\vec{B}_H e^{\kappa\zeta}\right) + \frac{i\omega\lambda_L^2 m^*c}{e^*}\vec{\nabla}\times(\vec{\nabla}\times\vec{h}) \tag{31}$$

Taking specific examples, the author then calculates the expressions for the fields for spherical geometries for both normal and super conductors. He finds the fields in the two cases to be of the same magnitude. The only difference being that the gauge invariant phase difference of the superelectrons has a bulk value

$$|\tilde{\theta}| \approx \frac{m^*c^2}{2\hbar\omega}(kR)^2 h$$

with $h \approx 10^{-24}$, $R \approx 1m$ and $\omega \approx 10^6 Hz$, we get $|\tilde{\theta}| \approx 10^{-14}$ which is small but possibly measurable.

He also calculates the total electromagnetically radiated power and the efficiency of conversion from gravitational to electromagnetic waves to be

$$\eta = \frac{40\pi G}{81\mu_0}\left(\frac{m}{ce}\right)^2(kR)^8 \approx 3\times10^{-44}(kR)^8 \tag{32}$$

which is very small even for meter sized spheres (here k is the wavenumber of the GW which is typically on the order of 10^{-6} m^{-1} .

Conclusion

The interaction between a gravitational wave and a superconductor was reviewed. A gravitational wave passing through a conductor was found to create Electric and Magnetic fields in the interior, but the magnitudes were found to be very small, and no significant difference was found between a normal and a superconductor except for a phase term, which could possibly be measurable.

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